

Supersymmetrization of Quaternion Dirac Equation for Generalized Fields of Dyons

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Abstract

The quaternion Dirac equation in presence of generalized electromagnetic field has been discussed in terms of two gauge potentials of dyons. Accordingly, the supersymmetry has been established consistently and thereafter the one, two and component Dirac Spinors of generalized quaternion Dirac equation of dyons for various energy and spin values are obtained for different cases in order to understand the duality invariance between the electric and magnetic constituents of dyons.

Key words: Supersymmetry, quaternion, Dirac equation, dyons

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1 Introduction:

Symmetries are one of the most powerful tools in the theoretical physics. Relativistic quantum mechanics is the theory of quantum mechanics that is consistent with the Einstein's theory of relativity. Dirac[1] was the first who attempted in this field followed by Feshback and Villars[2]. Since relativistic quantum mechanics in 3+1 space-time dimension becomes difficult because of different dimensionality of time and space. Nevertheless, the use of quaternions has become essential because quaternion algebra[3] has certain advantages. It provides 4-dimensional structure to relativistic quantum mechanics and also provide consistent representation in terms compact notations. Quaternions have direct link with Pauli spin matrices where the spin [4, 5] plays an important role in order to make connection between bosons and fermions. Pioneer work in the field of relativistic quaternionic quantum mechanics was done by Adler[4] while Rotelli[6] and Leo et al[7, 8] discussed the quaternionic wave equation. Gürsey[9] and Hestens[10] reformulated the Dirac equation from quaternion valued terms showing that the algebraic equivalent of Dirac has been forced to break the automorphism group of quaternions. Supersymmetric formulation of quaternionic quantum mechanics [4] has been discussed by Davies [11] into study supersymmetric quantum mechanics. More over, a lot of literature has been cited [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] to describe the supersymmetry (SUSY) as the natural symmetry of spin - particles. Nicolai [24] has also introduced the SUSY for spin system in statistical mechanics. Consequently, supersymmetric method in quaternionic Dirac equation provides [11] the exact solutions of various problems. Keeping in view the advantages of SUSY and the applications of quaternionic algebra, we [25, 26] have also analyzed the supersymmetrization of quaternion quantum mechanics and quaternion Dirac equation for different masses. Extending our results [26] , in this paper, we have discussed the quaternion Dirac equation in electromagnetic field where the partial derivative has been replaced by the quaternion covariant derivative. The quaternion Dirac equation in electromagnetic field consists of two gauge fields subjected by two unitary gauge transformations in terms of two gauge potentials. These two gauge potentials are identified as the gauge potentials respectively associated with the simultaneous existence of electric and magnetic charge (particles named as dyons [27, 28]). Accordingly, we have obtained the one and two components solutions of generalized quaternion Dirac equation of dyons for its different cases associated with its electric and magnetic constituents. Furthermore, we have analyzed, the supersymmetrization of generalized quaternion Dirac equation of dyons for considering different cases of electric and magnetic fields interacting with electric and magnetic charges as the consequence of electromagnetic duality of dyons.

2 Quaternion Preliminaries:

The algebra \mathbb{H} of quaternion is a four-dimensional algebra over the field of real numbers \mathbb{R} and a quaternion ϕ is expressed in terms of its four base elements as

$$\phi = \phi_\mu e_\mu = \phi_0 + e_1\phi_1 + e_2\phi_2 + e_3\phi_3 (\forall \mu = 0, 1, 2, 3) \quad (1)$$

where $\phi_0, \phi_1, \phi_2, \phi_3$ are the real quartets of a quaternion and e_0, e_1, e_2, e_3 are called quaternion units and satisfies the following relations,

$$\begin{aligned} e_0^2 &= e_0 = 1,; \quad e_j^2 = -e_0; \\ e_0 e_i &= e_i e_0 = e_i (i = 1, 2, 3); \\ e_i e_j &= -\delta_{ij} + \varepsilon_{ijk} e_k (\forall i, j, k = 1, 2, 3) \end{aligned} \quad (2)$$

where δ_{ij} is the delta symbol and ε_{ijk} is the Levi Civita three index symbol having value ($\varepsilon_{ijk} = +1$) for cyclic permutation, ($\varepsilon_{ijk} = -1$) for anti cyclic permutation and ($\varepsilon_{ijk} = 0$) for any two repeated indices. Addition and multiplication are defined by the usual distribution law ($e_j e_k e_l = e_j (e_k e_l)$) along with the multiplication rules given by equation (2). \mathbb{H} is an associative but non commutative algebra. If $\phi_0, \phi_1, \phi_2, \phi_3$ are taken as complex quantities, the quaternion is said to be a bi- quaternion. Alternatively, a quaternion is defined as a two dimensional algebra over the field of complex numbers \mathbb{C} . We thus have $\phi = v + e_2 \omega (v, \omega \in \mathbb{C})$ and $v = \phi_0 + e_1 \phi_1$, $\omega = \phi_2 - e_1 \phi_3$ with the basic multiplication law changes to $ve_2 = -e_2 \bar{v}$. The quaternion conjugate $\bar{\phi}$ is defined as

$$\bar{\phi} = \phi_\mu \bar{e}_\mu = \phi_0 - e_1 \phi_1 - e_2 \phi_2 - e_3 \phi_3. \quad (3)$$

In practice ϕ is often represented as a 2×2 matrix $\phi = \phi_0 - i \vec{\sigma} \cdot \vec{\phi}$ where $e_0 = I$, $e_j = -i \sigma_j (j = 1, 2, 3)$ and σ_j are the usual Pauli spin matrices. Then $\bar{\phi} = \sigma_2 \phi^T \sigma_2$ with ϕ^T is the transpose of ϕ . The real part of the quaternion ϕ_0 is also defined as

$$Re \phi = \frac{1}{2}(\bar{\phi} + \phi) \quad (4)$$

where Re denotes the real part and if $Re \phi = 0$ then we have $\phi = -\bar{\phi}$ and imaginary ϕ is known as pure quaternion written as

$$\phi = e_1\phi_1 + e_2\phi_2 + e_3\phi_3. \quad (5)$$

The norm of a quaternion is expressed as $N(\phi) = \phi\bar{\phi} = \bar{\phi}\phi = \sum_{j=0}^3 \phi_j^2$ which is non negative i.e.

$$N(\phi) = |\phi| = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 = \text{Det.}(\phi) \geq 0. \quad (6)$$

Since there exists the norm of a quaternion, we have a division i.e. every ϕ has an inverse of a quaternion and is described as

$$\phi^{-1} = \frac{\bar{\phi}}{|\phi|}. \quad (7)$$

While the quaternion conjugation satisfies the following property

$$\overline{\phi_1\phi_2} = \overline{\phi_2}\overline{\phi_1}. \quad (8)$$

The norm of the quaternion (1) is positive definite and enjoys the composition law

$$N(\phi_1\phi_2) = N(\phi_1)N(\phi_2). \quad (9)$$

Quaternion (1) is also written as $\phi = (\phi_0, \vec{\phi})$ where $\vec{\phi} = e_1\phi_1 + e_2\phi_2 + e_3\phi_3$ is its vector part and ϕ_0 is its scalar part. So, the sum and product of two quaternions are described as

$$\begin{aligned} (\alpha_0, \vec{\alpha}) + (\beta_0, \vec{\beta}) &= (\alpha_0 + \beta_0, \vec{\alpha} + \vec{\beta}); \\ (\alpha_0, \vec{\alpha}) \cdot (\beta_0, \vec{\beta}) &= (\alpha_0\beta_0 - \vec{\alpha} \cdot \vec{\beta}, \alpha_0\vec{\beta} + \beta_0\vec{\alpha} + \vec{\alpha} \times \vec{\beta}). \end{aligned} \quad (10)$$

Quaternion elements are non-Abelian in nature and thus represent a non commutative division ring.

3 Quaternion Dirac Equation For Dyons:

The free particle quaternion Dirac equation is described [6] as,

$$(i\gamma^\mu \partial_\mu - m)\Psi(x, t) = 0 \quad (11)$$

where $\Psi(x, t) = \begin{pmatrix} \Psi_a(x, t) \\ \Psi_b(x, t) \end{pmatrix}$ is the two component spinor and

$$\Psi_a(x, t) = \Psi_0 + e_1 \Psi_1; \Psi_b(x, t) = \Psi_2 - e_1 \Psi_3 \quad (12)$$

are the components of spinor quaternion $\Psi = \Psi_0 + e_1 \Psi_1 + e_2 \Psi_2 + e_3 \Psi_3$ and Dirac γ matrices are also expressed in terms of quaternion units i.e.

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad \gamma_j = ie_j \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (\forall j = 1, 2, 3). \quad (13)$$

So, the set of pure quaternion field (1) remains invariant under the transformations

$$\phi \rightarrow \phi' = U\phi\bar{U}, \quad U \in Q, \quad U\bar{U} = 1. \quad (14)$$

Similarly, the quaternion conjugate $\bar{\phi}$ transforms as

$$\bar{\phi}' = \overline{U\phi\bar{U}} = U\overline{\phi\bar{U}} = -U\phi\bar{U} = -\phi' \quad (15)$$

Any $U \in Q$ has a decomposition like equation (14) which gives rise to a set $\{U \in Q; U\bar{U} = 1\} \sim SP(1) \sim SU(2)$. Though it has been emphasized earlier [4] that the automorphic transformation of Q -fields are local but one is free to select them according to the representations. On the other hand, a Q -field is subjected to more general $SO(4)$ transformations as

$$\phi \rightarrow \phi' = U_1\phi\bar{U}_2, \quad U_1, U_2 \in Q, \quad U_1\bar{U}_1 = U_2\bar{U}_2 = 1. \quad (16)$$

So, the covariant derivative may then be described [4] in terms of two Q -gauge fields i.e

$$D_\mu \phi = \partial_\mu \phi + A_\mu \phi - \phi B_\mu \quad (17)$$

which is subjected by two gauges A_μ and B_μ transforming like

$$\begin{aligned} A'_{\mu} &= U_1 A_{\mu} \overline{U_1} + (\partial_{\mu} U_1) \overline{U_1}; \\ B'_{\mu} &= U_2 B_{\mu} \overline{U_2} + (\partial_{\mu} U_2) \overline{U_2}; \end{aligned} \quad (18)$$

where A_{μ} and B_{μ} may be identified as the four potentials associated, respectively, with the electric and magnetic charges of dyons in terms of $U(1) \times U(1)$ gauge theory [28]. Here, the gauge transformations are Abelian and global. The quaternion covariant derivative given by equation (17) thus supports the idea of two four potentials of dyons. Accordingly, we may write the Dirac equation (11) for dyons on replacing the partial derivative ∂_{μ} by covariant derivative D_{μ} as

$$(i \gamma^{\mu} D_{\mu} - m) \Psi(x, t) = 0 \quad (19)$$

where the commutator is defined as

$$[D_{\mu}, D_{\nu}] \Psi = D_{\mu}(D_{\nu} \Psi) - D_{\nu}(D_{\mu} \Psi) = F_{\mu\nu} \Psi - \Psi \widetilde{F}_{\mu\nu}. \quad (20)$$

Here the gauge field strengths $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$ are described [28] as the generalized anti-symmetric dual invariant electromagnetic field tensors for dyons and are expressed as

$$\begin{aligned} F_{\mu\nu} &= \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu} - \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} (\partial^{\lambda} B^{\sigma} - \partial^{\sigma} B^{\lambda}); \\ \widetilde{F}_{\mu\nu} &= \partial_{\nu} B_{\mu} - \partial_{\mu} B_{\nu} - \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} (\partial^{\lambda} A^{\sigma} - \partial^{\sigma} A^{\lambda}); \end{aligned} \quad (21)$$

which leads to the following expressions [28] for the generalized electromagnetic fields of dyons i.e.

$$\begin{aligned} \vec{E} &= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi - \vec{\nabla} \times \vec{B}; \\ \vec{B} &= -\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \varphi + \vec{\nabla} \times \vec{A}; \end{aligned} \quad (22)$$

where $\{A_{\mu}\} = \{\phi, -\vec{A}\}$ and $\{B_{\mu}\} = \{\varphi, -\vec{B}\}$. Generalized electromagnetic field tensors (21) of dyons satisfy the following famous covariant form of Generalized Dirac-Maxwell's (GDM) equations in presence of magnetic monopoles[1] i.e.

$$\begin{aligned} F_{\mu\nu,\nu} &= j_\mu; \\ \widetilde{F_{\mu\nu,\nu}} &= k_\mu; \end{aligned} \quad (23)$$

where $\{j_\mu\} = \{\rho, -\vec{j}\} = \mathbf{e} \Psi \bar{\gamma}_\mu \Psi$ and $\{k_\mu\} = \{\varrho, -\vec{k}\} = \mathbf{g} \Psi \bar{\gamma}_\mu \Psi$ are described [28] as the four currents respectively associated with the electric \mathbf{e} and magnetic \mathbf{g} charges of dyons. We may now expand the four potentials (gauge potentials) in terms of quaternion as

$$\begin{aligned} A_\mu &= A_\mu^0 e_0 + A_\mu^1 e_1 + A_\mu^2 e_2 + A_\mu^3 e_3; \\ B_\mu &= B_\mu^0 e_0 + B_\mu^1 e_1 + B_\mu^2 e_2 + B_\mu^3 e_3. \end{aligned} \quad (24)$$

As such, the Abelian theory of dyons can now be restored by taking the real part of the quaternion (24) $A_\mu = \overline{A_\mu}$ and $B_\mu = \overline{B_\mu}$ implying that $(A_\mu^0)' = (A_\mu^0) = A_\mu$ and $(B_\mu^0)' = (B_\mu^0) = B_\mu$. However, if we consider the imaginary quaternion i.e. $A_\mu = -\overline{A_\mu}$ and $B_\mu = -\overline{B_\mu}$ we have the $SU(2) \times SU(2)$ gauge structure where $A_\mu = A_\mu^a e_a = A_\mu^1 e_1 + A_\mu^2 e_2 + A_\mu^3 e_3$ and $B_\mu = B_\mu^a e_a = B_\mu^1 e_1 + B_\mu^2 e_2 + B_\mu^3 e_3$. Thus, with the implementation of condition $U_1 \overline{U_1} = U_2 \overline{U_2} = 1$ there are only the six gauge fields A_μ^a and B_μ^a associated with the covariant derivative of Dirac equation (19). The transformation equation (16) is continuous and isomorphic to $SO(4)$ i.e.

$$\overline{\phi}' \phi' = \overline{(U_1 \phi \overline{U_2})} (U_1 \phi \overline{U_2}) = U_2 \overline{\phi} \overline{U_1} U_1 \phi \overline{U_2} = U_2 \overline{\phi} \phi \overline{U_2} = \overline{\phi} \phi. \quad (25)$$

The resulting Q –gauge theory has the correspondence $SO(4) \sim SO(3) \times SO(3)$ isomorphic to $SU(2) \times SU(2)$. Accordingly, the spinor transforms as left and right component (electric or magnetic) spinors as

$$\Psi_e \mapsto (\Psi_e)' = U_1 \Psi_e \quad \& \quad \Psi_g \mapsto (\Psi_g)' = U_2 \Psi_g. \quad (26)$$

The following split basis of quaternion units may also be considered as

$$\begin{aligned} u_0 &= \frac{1}{2}(1 - i e_3); \quad u_0^* = \frac{1}{2}(1 + i e_3); \\ u_1 &= \frac{1}{2}(e_1 + i e_2); \quad u_1^* = \frac{1}{2}(e_1 - i e_2); \end{aligned} \quad (27)$$

to constitute the $SU(2)$ doublets. As such, we may express the Q -classes into five groups and can expand the theory with these choices. These five irreducible representations of $SO(4)$ are realized as

$$\begin{aligned}
1. (U_1, U_2) &\Rightarrow SO(4) \mapsto (2, 2) \\
2. (U_1, U_1) &\Rightarrow SU(2) \mapsto (3, 1) \\
3. (U_2, U_2) &\Rightarrow SU(2) \mapsto (1, 3) \\
4. (U_1, 1) &\Rightarrow \text{Spinor} \mapsto (2, 1) \\
5. (U_2, 1) &\Rightarrow \text{Spinor} \mapsto (1, 2).
\end{aligned} \tag{28}$$

Accordingly, it is easier to develop a non-Abelian gauge theory of dyons. It is to be mentioned that the occurrence of two gauge potentials supports the idea of duality invariance [29] among the electric and magnetic parameters of dyons.

3 Supersymmetrization of Quaternion Dirac Equation for Dyons

Quaternion Dirac equation (11) for dyons may now be written as

$$i\gamma_\mu D_\mu \psi(x, t) = m\psi(x, t) \tag{29}$$

where γ matrices satisfy the properties

$$\begin{aligned}
\gamma_0^2 &= +1; \quad \gamma_l^2 = -1 \quad (\forall l = 1, 2, 3) \\
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu &= -2g_{\mu\nu} \quad (g_{\mu\nu} = -1, +1, +1, +1)
\end{aligned} \tag{30}$$

showing that γ_0 is Hermitian while γ_l are anti-Hermitian matrices. Accordingly, the matrix γ_5 may be expressed as

$$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{31}$$

which satisfies the relations

$$\begin{aligned}\gamma_0\gamma_5 + \gamma_5\gamma_0 &= 0; \\ \gamma_l\gamma_5 + \gamma_5\gamma_l &= 0; \quad \gamma_5^2 = -1.\end{aligned}\tag{32}$$

It shows that the matrix γ_5 is pseudo scalar matrix. Furthermore, the quaternionic Dirac spinor $\psi = \psi_0 + e_1\psi_1 + e_2\psi_2 + e_3\psi_3$ can now be decomposed as

$$\psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ -\psi_3 \end{pmatrix}\tag{33}$$

in terms of two and four components Dirac spinors associated with symplectic representation of quaternions $\psi = \psi_a + e_2\psi_b$ with $\psi_a = \psi_0 + e_1\psi_1$ and $\psi_b = \psi_2 - e_1\psi_3$. Furthermore, we may also write one component quaternion valued Dirac spinor which is isomorphic to two component complex spinor and four component real spinor representation. Substituting the value of D_μ from equation (1747), we get

$$i\gamma_\mu (\partial_\mu\psi(x, t) + \mathbf{e}A_\mu\psi(x, t) - \mathbf{g}\psi(x, t)B_\mu) = m\psi(x, t).\tag{34}$$

Splitting γ_μ , ∂_μ , A_μ and B_μ in terms of real and quaternionic constituents, we get

$$i\gamma_0 (\partial_0\psi + \mathbf{e}A_0\psi - \mathbf{g}\psi B_0) + i\gamma_l (\partial_l\psi + \mathbf{e}A_l\psi - \mathbf{g}\psi B_l) = m\psi;\tag{35}$$

which is the general equation of spin- $\frac{1}{2}$ particle (dyon) in generalized electromagnetic field. Equation (35) may now be reduced as

$$i\gamma_0 (-iE\psi + \mathbf{e}A_0\psi - \mathbf{g}\psi B_0) + i\gamma_l (ip_l\psi + \mathbf{e}e_lA_l\psi - \mathbf{g}\psi e_lB_l) = m\psi\tag{36}$$

which can also be written explicitly as

$$\begin{aligned}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} (E\psi + i\mathbf{e}A_0\psi - i\mathbf{g}\psi B_0) + \\ \begin{bmatrix} 0 & ie_l \\ -ie_l & 0 \end{bmatrix} (-P_l\psi + i\mathbf{e}e_lA_l\psi - i\mathbf{g}\psi e_lB_l) - m\psi = 0\end{aligned}\tag{37}$$

Let us study the above equation for different cases

3.1 Case (a) For electric field due to electric charge

Let us discuss the case when we have only pure electric field associated with electric charge \mathbf{e} . In this case we have $A_0 \neq 0$, $A_l = 0$, $B_\mu = 0$ so that the equation (37) reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} (E + i \mathbf{e} A_0) \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} + \begin{bmatrix} 0 & ie_l \cdot P_l \\ -ie_l \cdot p_l & 0 \end{bmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} - m \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = 0 \quad (38)$$

which further reproduces two coupled equations

$$\begin{aligned} \widehat{\mathcal{A}}^\dagger \psi_b &= ie_l \cdot P_l \psi_b = (E + i \mathbf{e} A_0 - m) \psi_a; \\ \widehat{\mathcal{A}} \psi_a &= ie_l \cdot P_l \psi_a = (E + i \mathbf{e} A_0 + m) \psi_b; \end{aligned} \quad (39)$$

where $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}^\dagger = ie_l p_l$. These two-coupled equations (39) can now be decoupled into a single equation leading to its supersymmetrization as

$$P_l^2 \psi_{a,b} = \{(E + i \mathbf{e} A_0)^2 - m^2\} \psi_{a,b} \quad (40)$$

so that the super partner Hamiltonian may now be written as

$$\begin{aligned} \widehat{\mathcal{H}}_- &= \widehat{\mathcal{A}}^\dagger \widehat{\mathcal{A}} = P_l^2; \\ \widehat{\mathcal{H}}_+ &= \widehat{\mathcal{A}} \widehat{\mathcal{A}}^\dagger = P_l^2. \end{aligned} \quad (41)$$

Corresponding Dirac Hamiltonian may be defined in the following manner where we have used the Pauli-Dirac representation i.e.

$$\widehat{\mathcal{H}}_D = \begin{bmatrix} m & ie_l P_l \\ ie_l P_l & -m \end{bmatrix}. \quad (42)$$

Let us write equation (42) as compared to the standard Dirac Hamiltonian given by Thaller [21] as

$$\widehat{\mathcal{H}}_D = \begin{bmatrix} M_+ & \hat{Q}_D^\dagger \\ \hat{Q}_D & M_- \end{bmatrix} \quad (43)$$

which leads to $M_+ = M_- = 0$ and $\hat{Q}_D = \hat{Q}_D^+ = ie_l P_l$ along with the following supersymmetric conditions

$$\begin{aligned}\hat{Q}_D^\dagger M_- &= M_+ \hat{Q}_D^\dagger; \\ \hat{Q}_D M_+ &= M_- \hat{Q}_D\end{aligned}\quad (44)$$

and the following expression for the square of the Dirac Hamiltonian i.e.

$$\hat{\mathcal{H}}_D^2 = \begin{bmatrix} (P_l^2 + m^2) & 0 \\ 0 & (P_l^2 + m^2) \end{bmatrix}. \quad (45)$$

As such, we may write the Schrodinger Hamiltonian \hat{H}_s and Supercharges \hat{Q}_s and \hat{Q}_s^\dagger as

$$\begin{aligned}\hat{H}_s &= \begin{bmatrix} P_l^2 & 0 \\ 0 & P_l^2 \end{bmatrix}; \\ \hat{Q}_s &= \begin{bmatrix} 0 & ie_l P_l \\ 0 & 0 \end{bmatrix}; \\ \hat{Q}_s^\dagger &= \begin{bmatrix} 0 & 0 \\ ie_l P_l & 0 \end{bmatrix};\end{aligned}\quad (46)$$

which satisfy the following well known forms of supersymmetric (SUSY) algebra i.e.

$$\begin{aligned}[\hat{Q}_s, \hat{H}_s] &= [\hat{Q}_s^\dagger, \hat{H}_s] = 0 \\ \{\hat{Q}_s, \hat{Q}_s\} &= \{\hat{Q}_s^\dagger, \hat{Q}_s^\dagger\} = 0 \\ [\hat{Q}_s, \hat{Q}_s^\dagger] &= \hat{H}_s^+.\end{aligned}\quad (47)$$

We may also obtain the following types of four spinor amplitudes of Dirac spinors i.e.

- One component spinor amplitudes

$$\begin{aligned}\Psi^1 &= (1 + e_2 \cdot \frac{ie_l P_l}{E_+ - \mathbf{e} A_0 + m}) \quad (Energy = +ive, spin = \uparrow); \\ \Psi^2 &= (1 + e_2 \cdot \frac{ie_l P_l}{E_+ - \mathbf{e} A_0 + m}) e_1 \quad (Energy = +ive, spin = \downarrow); \\ \Psi^3 &= (e_2 - \frac{ie_l P_l}{E_- + \mathbf{e} A_0 + m}) \quad (Energy = -ive, spin = \uparrow); \\ \Psi^4 &= (e_2 - \frac{ie_l P_l}{E_- + \mathbf{e} A_0 + m}) e_1 \quad (Energy = -ive, spin = \downarrow).\end{aligned}\quad (48)$$

- Two component spinor amplitudes

$$\begin{aligned}
\Psi^1 &= \begin{pmatrix} 1 \\ \frac{ie_l P_l}{E_+ - \mathbf{e} A_0 + m} \end{pmatrix} (\text{Energy} = +ive, \text{spin} = \uparrow); \\
\Psi^2 &= \begin{pmatrix} 1 \\ \frac{ie_l P_l}{E_+ - \mathbf{e} A_0 + m} \end{pmatrix} e_1 (\text{Energy} = +ive, \text{spin} = \downarrow); \\
\Psi^3 &= \begin{pmatrix} -\frac{ie_l P_l}{E_- + \mathbf{e} A_0 + m} \\ 1 \end{pmatrix} (\text{Energy} = -ive, \text{spin} = \uparrow); \\
\Psi^4 &= \begin{pmatrix} -\frac{ie_l P_l}{E_- + \mathbf{e} A_0 + m} \\ 1 \end{pmatrix} e_1 (\text{Energy} = -ive, \text{spin} = \downarrow). \tag{49}
\end{aligned}$$

- Four component spinor amplitudes may also be obtained by restricting the direction of propagation along any one axis which we suppose $Z-axis$ i.e $(p_x = p_y = 0)$ and on substituting $e_l = -i\sigma_l$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ along with the usual definitions of spin up and spin down amplitudes of spin i.e.

$$\begin{aligned}
\psi^1 &= \begin{pmatrix} 1 \\ 0 \\ \frac{|\vec{p}|}{E_+ - \mathbf{e} A_0 + m} \\ 0 \end{pmatrix} (\text{Energy} = +ive, \text{spin} = \uparrow); \\
\psi^2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{|\vec{p}|}{E_+ - \mathbf{e} A_0 + m} \end{pmatrix} (\text{Energy} = +ive, \text{spin} = \downarrow); \\
\psi^3 &= \begin{pmatrix} -\frac{|\vec{p}|}{E_- + \mathbf{e} A_0 + m} \\ 0 \\ 1 \\ 0 \end{pmatrix} (\text{Energy} = -ive, \text{spin} = \uparrow); \\
\psi^4 &= \begin{pmatrix} 0 \\ \frac{|\vec{p}|}{E_- + \mathbf{e} A_0 + m} \\ 0 \\ 1 \end{pmatrix} (\text{Energy} = -ive, \text{spin} = \downarrow). \tag{50}
\end{aligned}$$

As such, we have obtained the solution of quaternion Dirac equation for dyons in terms of one component quaternion, two component complex and four component real spinor

amplitudes. Equation (50) is same as obtained for the case of usual Dirac equation in electromagnetic field. Thus we may interpret that the $N = 1$ quaternion spinor amplitude is isomorphic to $N = 2$ complex and $N = 4$ real spinor amplitude solution of Dirac equation for dyons. We can accordingly interpret the minimum dimensional representation for Dirac equation is $N = 1$ in quaternionic case, $N = 2$ in complex case and $N = 4$ for real number field.

3.2 Case (b): For magnetic field due to electric charge

Let us discuss the case when we have only pure magnetic associated with electric charge \mathbf{e} . In this case we have $A_0 = 0$, $A_l \neq 0$, $B_\mu = 0$ so that the equation (37) reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} E \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} + \begin{bmatrix} 0 & ie_l \\ -ie_l & 0 \end{bmatrix} (-P_l + i \mathbf{e} e_l A_l) \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} - m \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = 0 \quad (51)$$

which yields two coupled equations i.e.

$$\begin{aligned} \widehat{\mathcal{A}}^\dagger \psi_b &= ie_l (P_l - i \mathbf{e} e_l A_l) \psi_b = (E - m) \psi_a; \\ \widehat{\mathcal{A}} \psi_a &= ie_l (P_l - i \mathbf{e} e_l A_l) \psi_a = (E + m) \psi_b; \end{aligned} \quad (52)$$

where $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}^\dagger = ie_l (P_l - i e_l A_l)$. These two-coupled equations can be decoupled into a single coupled equation showing supersymmetry in the following manner

$$[ie_l (P_l - i \mathbf{e} e_l A_l)]^2 \psi_{a,b} = \{E - m^2\} \psi_{a,b} \quad (53)$$

so that the super partner Hamiltonian may now be written as

$$\widehat{\mathcal{H}}_- = \widehat{\mathcal{A}}^\dagger \widehat{\mathcal{A}} = \widehat{\mathcal{H}}_+ = \widehat{\mathcal{A}} \widehat{\mathcal{A}}^\dagger = [ie_l (P_l - i \mathbf{e} e_l A_l)]^2. \quad (54)$$

Thus the corresponding Dirac Hamiltonian may be defined in the following manner

$$\widehat{\mathcal{H}}_D = \begin{bmatrix} m & ie_l (P_l - i \mathbf{e} e_l A_l) \\ ie_l (P_l - i \mathbf{e} e_l A_l) & -m \end{bmatrix}. \quad (55)$$

Like wise, the previous case of electric field, here in case of magnetic field we may also obtain $M_+ = M_- = m$ and $\hat{Q}_D = \hat{Q}_D^+ = ie_l (P_l - i \mathbf{e} e_l A_l)$ along with the supersymmetric condition (44) and the following expression for the square of the Dirac Hamiltonian as

$$\hat{\mathcal{H}}_D^2 = \begin{bmatrix} [ie_l(P_l - i \mathbf{e} e_l A_l)]^2 + m^2 & 0 \\ 0 & [ie_l(P_l - i \mathbf{e} e_l A_l)]^2 + m^2 \end{bmatrix}. \quad (56)$$

Accordingly we may write the Schrodinger Hamiltonian \hat{H}_s and Supercharges \hat{Q}_s and \hat{Q}_s^\dagger as

$$\begin{aligned} \hat{H}_s &= \begin{bmatrix} [ie_l(P_l - i \mathbf{e} e_l A_l)]^2 & 0 \\ 0 & [ie_l(P_l - i \mathbf{e} e_l A_l)]^2 \end{bmatrix}; \\ \hat{Q}_s &= \begin{bmatrix} 0 & [ie_l(P_l - i \mathbf{e} e_l A_l)] \\ 0 & 0 \end{bmatrix}; \\ \hat{Q}_s^\dagger &= \begin{bmatrix} 0 & 0 \\ [ie_l(P_l - i \mathbf{e} e_l A_l)] & 0 \end{bmatrix}. \end{aligned} \quad (57)$$

Here, also \hat{H}_s , \hat{Q}_s and \hat{Q}_s^\dagger satisfy the well known supersymmetric (SUSY) algebra given by equation (47). Consequently, we may also obtain the following types of four spinor amplitudes of Dirac spinors in presence of pure magnetic field as i.e.

- One component spinor amplitudes

$$\begin{aligned} \Psi^1 &= (1 + e_2 \cdot \frac{[ie_l(P_l - i \mathbf{e} e_l A_l)]}{E_+ + m}) \quad (Energy = +ive, spin = \uparrow); \\ \Psi^2 &= (1 + e_2 \cdot \frac{[ie_l(P_l - i \mathbf{e} e_l A_l)]}{E_+ + m}) e_1 \quad (Energy = +ive, spin = \downarrow); \\ \Psi^3 &= (e_2 - \frac{[ie_l(P_l - i \mathbf{e} e_l A_l)]}{E_- + m}) \quad (Energy = -ive, spin = \uparrow); \\ \Psi^4 &= (e_2 - \frac{[ie_l(P_l - i \mathbf{e} e_l A_l)]}{E_- + m}) e_1 \quad (Energy = -ive, spin = \downarrow). \end{aligned} \quad (58)$$

- Two component spinor amplitudes

$$\begin{aligned} \Psi^1 &= \begin{pmatrix} 1 \\ \frac{[ie_l(P_l - i \mathbf{e} e_l A_l)]}{E_+ + m} \end{pmatrix} (Energy = +ive, spin = \uparrow); \\ \Psi^2 &= \begin{pmatrix} 1 \\ \frac{[ie_l(P_l - i \mathbf{e} e_l A_l)]}{E_+ + m} \end{pmatrix} e_1 (Energy = +ive, spin = \downarrow); \\ \Psi^3 &= \begin{pmatrix} -\frac{[ie_l(P_l - i \mathbf{e} e_l A_l)]}{E_- + m} \\ 1 \end{pmatrix} (Energy = -ive, spin = \uparrow); \\ \Psi^4 &= \begin{pmatrix} -\frac{[ie_l(P_l - i \mathbf{e} e_l A_l)]}{E_- + m} \\ 1 \end{pmatrix} e_1 (Energy = -ive, spin = \downarrow). \end{aligned} \quad (59)$$

- Four component spinor amplitudes may also be obtained by restricting the direction of propagation along any one axis which we suppose $Z - axis$ i.e $(p_x = p_y = 0)$ and $(A_x = A_y = 0 \Rightarrow H_z = 0)$. Accordingly, substituting $e_l = -i\sigma_l$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ along with the usual definitions of spin up and spin down amplitudes of spin, we get

$$\begin{aligned}
\psi^1 &= \begin{pmatrix} 1 \\ 0 \\ \frac{|\vec{p}|}{E_+ + m} \\ 0 \end{pmatrix} \text{ (Energy = +ive, spin = \uparrow);} \\
\psi^2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{|\vec{p}|}{E_+ + m} \end{pmatrix} \text{ (Energy = +ive, spin = \downarrow);} \\
\psi^3 &= \begin{pmatrix} -\frac{|\vec{p}|}{E_- + m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ (Energy = -ive, spin = \uparrow);} \\
\psi^4 &= \begin{pmatrix} 0 \\ \frac{|\vec{p}|}{E_- + m} \\ 0 \\ 1 \end{pmatrix} \text{ (Energy = -ive, spin = \downarrow).} \tag{60}
\end{aligned}$$

which are the well known usual spinor amplitudes for a Dirac free Particle .

3.3 Case (c): For Electric field due to magnetic monopole

Here, we discuss the case when we have only electric field associated with magnetic charge (pure magnetic monopole) \mathbf{g} only. So, by virtue of duality of magnetic charge [27, 28, 29, 30], we take $B_0 = 0$, $B_l \neq 0$, $A_\mu = 0$. Thus, the equation (37) reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} E \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} + \begin{bmatrix} 0 & ie_l \\ -ie_l & 0 \end{bmatrix} (-P_l + i \mathbf{g} e_l B_l) \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} - m \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = 0 \tag{61}$$

which yields two coupled equations i.e.

$$\begin{aligned}\hat{\mathcal{A}}^\dagger \psi_b &= ie_l(P_l - i \mathbf{g} e_l B_l) \psi_b = (E - m) \psi_a; \\ \hat{\mathcal{A}} \psi_a &= ie_l(P_l - i \mathbf{g} e_l B_l) \psi_a = (E + m) \psi_b;\end{aligned}\quad (62)$$

where $\hat{\mathcal{A}} = \hat{\mathcal{A}}^\dagger = ie_l(P_l - i \mathbf{g} e_l B_l)$. These two-coupled equations can be decoupled into a single coupled equation showing supersymmetry in the following manner

$$[ie_l(P_l - i \mathbf{g} e_l B_l)]^2 \psi_{a,b} = \{E - m^2\} \psi_{a,b} \quad (63)$$

so that the super partner Hamiltonian may now be written as

$$\hat{\mathcal{H}}_- = \hat{\mathcal{A}}^\dagger \hat{\mathcal{A}} = \hat{\mathcal{H}}_+ = \hat{\mathcal{A}} \hat{\mathcal{A}}^\dagger = [ie_l(P_l - i \mathbf{g} e_l B_l)]^2. \quad (64)$$

Thus the corresponding Dirac Hamiltonian may be defined in the following manner

$$\hat{\mathcal{H}}_D = \begin{bmatrix} m & ie_l(P_l - i \mathbf{g} e_l B_l) \\ ie_l(P_l - i \mathbf{g} e_l B_l) & -m \end{bmatrix}. \quad (65)$$

Like wise, the previous case of electric field, here in case of magnetic field we may also obtain $M_+ = M_- = m$ and $\hat{Q}_D = \hat{Q}_D^+ = ie_l(P_l - i \mathbf{g} e_l B_l)$ along with the supersymmetric condition (44) and the following expression for the square of the Dirac Hamiltonian as

$$\hat{\mathcal{H}}_D^2 = \begin{bmatrix} [ie_l(P_l - i \mathbf{g} e_l B_l)]^2 + m^2 & 0 \\ 0 & [ie_l(P_l - i \mathbf{g} e_l B_l)]^2 + m^2 \end{bmatrix}. \quad (66)$$

Accordingly we may write the Schrodinger Hamiltonian \hat{H}_s and Supercharges \hat{Q}_s and \hat{Q}_s^\dagger as

$$\begin{aligned}\hat{H}_s &= \begin{bmatrix} [ie_l(P_l - i \mathbf{g} e_l B_l)]^2 & 0 \\ 0 & [ie_l(P_l - i \mathbf{g} e_l B_l)]^2 \end{bmatrix}; \\ \hat{Q}_s &= \begin{bmatrix} 0 & [ie_l(P_l - i \mathbf{g} e_l B_l)] \\ 0 & 0 \end{bmatrix}; \\ \hat{Q}_s^\dagger &= \begin{bmatrix} 0 & 0 \\ [ie_l(P_l - i \mathbf{g} e_l B_l)] & 0 \end{bmatrix}.\end{aligned}\quad (67)$$

Here, also \hat{H}_s , \hat{Q}_s and \hat{Q}_s^\dagger satisfy the well known supersymmetric (SUSY) algebra given by equation (47). Consequently, we may also obtain the following types of four spinor amplitudes of Dirac spinors in presence of pure magnetic field as i.e.

- One component spinor amplitudes

$$\begin{aligned}
 \Psi^1 &= (1 + e_2 \cdot \frac{[ie_l(P_l - i\mathbf{g}e_lB_l)]}{E_+ + m}) \quad (Energy = +ive, spin = \uparrow); \\
 \Psi^2 &= (1 + e_2 \cdot \frac{[ie_l(P_l - i\mathbf{g}e_lB_l)]}{E_+ + m})e_1 \quad (Energy = +ive, spin = \downarrow); \\
 \Psi^3 &= (e_2 - \frac{[ie_l(P_l - i\mathbf{g}e_lB_l)]}{E_- + m}) \quad (Energy = -ive, spin = \uparrow); \\
 \Psi^4 &= (e_2 - \frac{[ie_l(P_l - i\mathbf{g}e_lB_l)]}{E_- + m})e_1 \quad (Energy = -ive, spin = \downarrow).
 \end{aligned} \tag{68}$$

- Two component spinor amplitudes

$$\begin{aligned}
 \Psi^1 &= \begin{pmatrix} 1 \\ \frac{[ie_l(P_l - i\mathbf{g}e_lB_l)]}{E_+ + m} \end{pmatrix} (Energy = +ive, spin = \uparrow); \\
 \Psi^2 &= \begin{pmatrix} 1 \\ \frac{[ie_l(P_l - i\mathbf{g}e_lB_l)]}{E_+ + m} \end{pmatrix} e_1 (Energy = +ive, spin = \downarrow); \\
 \Psi^3 &= \begin{pmatrix} -\frac{[ie_l(P_l - i\mathbf{e}e_lA_l)]}{E_- + m} \\ 1 \end{pmatrix} (Energy = -ive, spin = \uparrow); \\
 \Psi^4 &= \begin{pmatrix} -\frac{[ie_l(P_l - i\mathbf{g}e_lB_l)]}{E_- + m} \\ 1 \end{pmatrix} e_1 (Energy = -ive, spin = \downarrow).
 \end{aligned} \tag{69}$$

- Four component spinor amplitudes may also be obtained by restricting the direction of propagation along any one axis which we suppose $Z - axis$ i.e $(p_x = p_y = 0)$ and $(B_x = B_y = 0 \Rightarrow E_z = 0)$. Accordingly, substituting $e_l = -i\sigma_l$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ along with the usual definitions of spin up and spin down amplitudes of spin, we get the four component Dirac spinors same as the equation (60).

3.4 Case (d): For Magnetic field due to magnetic monopole

Let us discuss the case when we have only pure magnetic field associated with magnetic charge (monopole) \mathbf{g} . In this case we have $B_0 \neq 0$, $B_l = 0$, $A_\mu = 0$ so that the equation (37) reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} (E + i \mathbf{g} B_0) \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} + \begin{bmatrix} 0 & ie_l P_l \\ -ie_l p_l & 0 \end{bmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} - m \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = 0 \quad (70)$$

which further reduces to two coupled equations

$$\begin{aligned} \widehat{\mathcal{A}}^\dagger \psi_b &= ie_l P_l \psi_b = (E + i \mathbf{g} B_0 - m) \psi_a; \\ \widehat{\mathcal{A}} \psi_a &= ie_l P_l \psi_a = (E + i \mathbf{g} B_0 - m) \psi_b; \end{aligned} \quad (71)$$

where $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}^\dagger = ie_l p_l$. These two-coupled equations (71) can now be decoupled into a single equation leading to its supersymmetrization as

$$P_l^2 \psi_{a,b} = \{(E + i \mathbf{g} B_0)^2 - m^2\} \psi_{a,b}; \quad (72)$$

so that the super partner Hamiltonian may now be written as equation (41). Corresponding Dirac Hamiltonian then may be defined as equation (42) which can also be written as (43) after its comparison with the standard Dirac Hamiltonian given by Thaller [21] and thus, leads to $M_+ = M_- = 0$ and $\hat{Q}_D = \hat{Q}_D^+ = ie_l P_l$ along with the following supersymmetric conditions given by equation (44) along with the Dirac Hamiltonian given by

$$\widehat{\mathcal{H}}_D^2 = \begin{bmatrix} (P_l^2 + m^2) & 0 \\ 0 & (P_l^2 + m^2) \end{bmatrix} = \hat{H}_s^2 + m^2 \hat{I} \quad (73)$$

where \hat{I} is unit matrix of order 4. Consequently, we may write the Schrodinger Hamiltonian \hat{H}_s and Supercharges (\hat{Q}_s and \hat{Q}_s^\dagger) as given by equation (46) leading to well known supersymmetric (SUSY) algebra relations given by equation (47). Furthermore, the following types of four spinor amplitudes of Dirac spinors may also be obtained as

- One component spinor amplitudes

$$\begin{aligned} \Psi^1 &= (1 + e_2 \cdot \frac{ie_l P_l}{(E_+ - \mathbf{g} B_0 + m)}) \quad (Energy = +ive, spin = \uparrow); \\ \Psi^2 &= (1 + e_2 \cdot \frac{ie_l P_l}{(E_+ - \mathbf{g} B_0 + m)}) e_1 \quad (Energy = +ive, spin = \downarrow); \\ \Psi^3 &= (e_2 - \frac{ie_l P_l}{(E_- + \mathbf{g} B_0 + m)}) \quad (Energy = -ive, spin = \uparrow); \\ \Psi^4 &= (e_2 - \frac{ie_l P_l}{(E_- + i \mathbf{g} B_0 + m)}) e_1 \quad (Energy = -ive, spin = \downarrow). \end{aligned} \quad (74)$$

- Two component spinor amplitudes

$$\begin{aligned}
\Psi^1 &= \begin{pmatrix} 1 \\ \frac{ie_l P_l}{(E_+ - \mathbf{g} B_0 + m)} \end{pmatrix} (\text{Energy} = +ive, \text{spin} = \uparrow); \\
\Psi^2 &= \begin{pmatrix} 1 \\ \frac{ie_l P_l}{(E_+ - \mathbf{g} B_0 + m)} \end{pmatrix} e_1 (\text{Energy} = +ive, \text{spin} = \downarrow); \\
\Psi^3 &= \begin{pmatrix} -\frac{ie_l P_l}{(E_- + \mathbf{g} B_0 + m)} \\ 1 \end{pmatrix} (\text{Energy} = -ive, \text{spin} = \uparrow); \\
\Psi^4 &= \begin{pmatrix} -\frac{ie_l P_l}{(E_- + \mathbf{g} B_0 + m)} \\ 1 \end{pmatrix} e_1 (\text{Energy} = -ive, \text{spin} = \downarrow). \tag{75}
\end{aligned}$$

- Four component spinor amplitudes may also be obtained by restricting the direction of propagation along any one axis which we suppose $Z-axis$ i.e $(p_x = p_y = 0)$ and on substituting $e_l = -i\sigma_l$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ along with the usual definitions of spin up and spin down amplitudes of spin i.e.

$$\begin{aligned}
\psi^1 &= \begin{pmatrix} 1 \\ 0 \\ \frac{|\vec{p}|}{(E_+ - \mathbf{g} B_0 + m)} \\ 0 \end{pmatrix} (\text{Energy} = +ive, \text{spin} = \uparrow); \\
\psi^2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{|\vec{p}|}{(E_+ - \mathbf{g} B_0 + m)} \end{pmatrix} (\text{Energy} = +ive, \text{spin} = \downarrow); \\
\psi^3 &= \begin{pmatrix} -\frac{|\vec{p}|}{(E_- + \mathbf{g} B_0 + m)} \\ 0 \\ 1 \\ 0 \end{pmatrix} (\text{Energy} = -ive, \text{spin} = \uparrow); \\
\psi^4 &= \begin{pmatrix} 0 \\ \frac{|\vec{p}|}{(E_- + \mathbf{g} B_0 + m)} \\ 0 \\ 1 \end{pmatrix} (\text{Energy} = -ive, \text{spin} = \downarrow). \tag{76}
\end{aligned}$$

4 Discussion and Conclusion

We have discussed the quaternion Dirac equation in electromagnetic field where the partial derivative has been replaced by the quaternion covariant derivative in terms of two gauge potentials. These two gauge potentials are identified as the gauge potentials associated with a particle which contains the simultaneous existence of electric and magnetic charge (monopole). Such type of particles are named as dyons. Thereafter, we have established consistently the SUSY for different cases of quaternion Dirac equation for dyons. **Case (a)** deals with the study of supersymmetrization of Dirac equation when it interacts with electric field produced by electric charge only. Thereby, we have obtained a single decoupled equation, super partner Hamiltonians, total (Schrodinger) Hamiltonians and Schrodinger supercharges consistently followed by the Dirac Hamiltonian and Dirac supercharges. It is shown that the supercharges and Hamiltonian satisfy the SUSY algebra performing SUSY transformations. Moreover, in this case, we have also obtained the solutions of the Dirac equation for one component, two components and four component Dirac spinors with various energy and spins. **Case (b)** is described for a dyon consisting electric charge but moves in only magnetic field. Likewise, we have followed the same procedure and obtained consistently the particle Hamiltonian and supercharges to satisfy the SUSY algebra. Furthermore, we have obtained the consistently one component, two components and four component Dirac spinors with various energy and spins. Same procedure has also been extended for **Case (c)** and **Case (d)** respectively associated with the electric and magnetic fields due to the presence of magnetic monopole in order to establish the consistent formulation of SUSY and Dirac spinors of various energy and spins. It is concluded that the **Case (a)** and **Case (d)** and likewise, **Case (b)** and **Case (c)** are dual invariant. These cases may also be analyzed by applying the duality transformations between electric and magnetic constituents of dyons. It may also be concluded that minimal representation for quaternion Dirac equation is described as $N = 1$ quaternionic, $N = 2$ complex and $N = 4$ real representation. In fact, the one-component spinor amplitudes are isomorphic to two component complex spinor amplitudes and four component real spinor amplitudes. As such, the higher dimensional supersymmetric Dirac equation in generalized electromagnetic fields of dyons may be tackled well in terms of quaternions splitting into $N = 1$ quaternionic, $N = 2$ complex and $N = 4$ real representations of Supersymmetric quantum mechanics.

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References

- [1] P. A. M. Dirac, “**The principles of quantum mechanics**”, (4th ed) Oxford University Press London (1958).
- [2] H. Feshback and F. Villars, Rev. Mod. Phys., **30** (1958), 24.
- [3] J. Souček, J. Phys. A: Math.Gen., **14** (1981), 1629.
- [4] S. L. Adler; “**Quaternionic Quantum mechanics**”, Oxford University Press, Oxford (1995).
- [5] A. Das, S. Okubo and S.A. Pernice, Modern Physics Letters, **A12** (1997), 581.
- [6] P. Rotelli, Modern Phys letters, **4** (1989) 1763.
- [7] S. De Leo and P.Rotelli, Prog.Theor.Phys., **92** (1994), 917.
- [8] S. De Leo and P.Rotelli, Mod.Phys.Lett., **A11** (1996), 357.
- [9] F. Gürsey, Rev. fac. Sci., Univ. Istribul (Turkey), **A21** (1956), 33.
- [10] D. Hestens, J. Math. Phys., **8** (1967), 778.
- [11] A. J. Davies, Phys. Rev. **A49** (1994), 714.
- [12] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep., **251** (1995) 267.
- [13] F. Cooper, A. Khare and U. Sukhatme, Ann. Phys (NY), **187** (1988)1.
- [14] C.V. Sukumar, J. Phys., **A 18** (1985), 2917 & 2937.
- [15] L. P. Singh and B. Ram, Pramana-Journal of Physics; **58** (2002), 591
- [16] S. V.ketov and Ya S. Prager, Acta Phys. Pol., **B21** (1990) , 463.
- [17] T. E. Clark and S.T. Love, Nucl.Phys., **B231** (1984), 91.
- [18] M. de Crombrugghe and V. Rittenberg, Annals of Physics, **151** (1983), 99.
- [19] B. Thaller; J. Math. Phys ., **29**(1988), 247.
- [20] B. Thaller; “**Dirac particle in magnetic fields**”, in A.Boulet de Monrel, P. Dita, G. Nenciu and R.Purice Eds, ’**Recent Development in quantum mechanics; Mathematical Physics Studies**’ Nr.12 , Kluwer Acad. Publ. Dordrechel,(1991), pp.351-366.
- [21] B. Thaller; “**The Dirac equation**”, Springer Verlag, Berlin, (1992).

- [22] A. A. Andrianov, F. Cannata, J. P. Dedonder and M.V.Ioffe, Int. J. Mod. Phys., **A10** (1995), 2683
- [23] E. Witten, Nucl.Phys., **B188** (1981), 513.
- [24] H. Nicolai; J.Phys., **A 9** (1976), 1497.
- [25] Seema Rawat and O.P.S. Negi, Int. J. Theor. Phys., **48** (2009), 305.
- [26] Seema Rawat and O.P.S. Negi, Int. J. Theor. Phys., **48** (2009), 2222
- [27] P. S. Bisht and O. P. S. Negi, Int. J. Theor. Phys., **47** (2008), 1497.
- [28] O. P. S. Negi and H. Dehnen, Int. J. Theor. Phys., **50** (2011), 2446.
- [29] P. S. Bisht and O. P. S. Negi, Int. J. Theor. Phys., **47** (2008), 3108.
- [30] Y. M. Shnir, “**Magnetic Monopoles**”, Springer-Verlag Berlin-Heidelberg (2005).